Supplement to “Stationary Monetary Equilibria with Strictly Increasing Value Functions and Non-Discrete Money Holdings Distributions: An Indeterminacy Result”

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This note is a supplemental material to Kamiya and Shimizu [1] (hereafter, KS). We prove Theorems 2 and 3 in KS.

1 Proof of Theorem 2

Theorem 2 Suppose agents can hold any amount of money, i.e., $B = \infty$. Suppose $\frac{3}{2} < d \leq 3$. Let $\beta = \frac{3k}{3(k-1)+2d}$. Then, for any given $\beta \in (\beta, 1)$, there exists a continuum of stationary equilibria in which (i) the value functions are continuous, strictly increasing, and concave, and (ii) the money holdings distributions have a full support in some closed interval with a nonempty interior.

Proof:

(I) We extend the strategy and money holdings distribution constructed in the proof of Theorem 1 of KS to the environment without an upper bound of individual money holdings. In other words, for some $p > 0$,

- an agent without money always chooses to be a seller and an agent with money holding $\eta > 0$ always chooses to be a buyer,
- a seller always offers $(p, \bar{q})$,

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• a buyer with money holding $\eta > 0$ consumes the following amount of her consumption good: there exists a $p(\eta) \geq p$ such that, for given $(p_s, q_s)$,

$$q_b(\eta, p_s, q_s) = \begin{cases} \min\{\eta/p_s, q_s\} & \text{if } p_s \leq p(\eta), \\ 0 & \text{if } p_s > p(\eta), \end{cases}$$  \hspace{1cm} (1)

• for some $\lambda$ and $\sigma$, $f$ is expressed by

$$f(\eta) = \begin{cases} 2\lambda \eta + \sigma, & \text{for } \eta \in (0, \bar{p}q], \\ 0, & \text{for } \eta \in (\bar{p}q, \infty). \end{cases}$$  \hspace{1cm} (2)

Note that $p$, $p(\eta)$, $\lambda$, and $\sigma$ will be determined as functions of $m_0$ later.

(II) Next, we obtain a candidate for a value function $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ consistent with the above strategy. From the above strategy, $V(\eta)$ for $\eta \in (0, \infty)$ can be written as a function of $V(0)$ as follows. First, for $\eta \in (0, \bar{p}q]$,

$$V(\eta) = \frac{m_0}{k} \left( a\frac{\eta}{p} + \beta V(0) \right) + \left( 1 - \frac{m_0}{k} \right) \beta V(\eta)$$

holds. Thus

$$V(\eta) = A(m_0) \left( a\frac{\eta}{p} + \beta V(0) \right), \quad \text{for } \eta \in (0, \bar{p}q],$$

where $A(m_0) = \frac{m_0}{k - (k-m_0)\beta}$. Note that $A(m_0) < 1$. Similarly, $V(\eta)$ for $\eta \in (\bar{p}q, \infty)$ is written as:

$$V(\eta) = A(m_0) (a\bar{q} + \beta V(\eta - \bar{p}q)), \quad \text{for } \eta \in (\bar{p}q, \infty).$$  \hspace{1cm} (4)

Next, since an agent without money always chooses to be a seller, then $V(0)$ is determined by

$$V(0) = \frac{1 - m_0}{k} \left[ -\bar{c} + \int_{(0, \bar{p}q]} \beta V(\eta) \frac{f(\eta)}{1 - m_0} d\eta \right] + \left( 1 - \frac{1 - m_0}{k} \right) \beta V(0).$$  \hspace{1cm} (5)

(III) Below, we focus on equilibria with $V(0) = 0$ and obtain $(p, \lambda, \sigma)$ as functions of $m_0$. First, we decompose $\eta \geq 0$ into an multiple of $p\bar{q}$ and a residual; that is, $\eta = np\bar{q} + \iota$, where
\( n \) is a nonnegative integer and \( \iota \) is a nonnegative real number less than \( \bar{p} \bar{q} \). Then, by (3) and (4),
\[
V(n\bar{p} \bar{q} + \iota) = \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^{\bar{q}} \left[ \bar{q} - (1 - \beta A(m_0)) \frac{\iota}{\bar{p}} \right] \right\}
\]
holds. On the other hand, by (2) and (5),
\[
(1 - m_0)\bar{c} = \frac{a\beta A(m_0)}{p} \int_{(0, \bar{p} \bar{q}]} \eta f(\eta) d\eta = a\beta A(m_0) \left( \frac{2}{3} \lambda p^2 \bar{q}^2 + \frac{1}{2} \sigma \bar{p}^2 \bar{q}^2 \right)
\]
holds.

Below, we obtain \((p, \lambda, \sigma)\) as functions of \( m_0 \). First, \( 1 - m_0 = \int_{(0, \infty]} f d\eta \) can be written as follows:
\[
1 - m_0 = \int_{(0, \bar{p} \bar{q}]} f d\eta = \lambda p^2 \bar{q}^2 + \sigma \bar{p} \bar{q}.
\]
Since the total amount of money the agents have is equal to \( M \), the following equation must be satisfied:
\[
M = \int_{(0, \bar{p} \bar{q}]} \eta f d\eta = \frac{2}{3} \lambda p^2 \bar{q}^3 + \frac{1}{2} \sigma p^2 \bar{q}^2.
\]
By (7), (8), (9), and \( d = \frac{a \bar{q}}{\bar{c}} \), we obtain
\[
p = \frac{Ma\beta A(m_0)}{(1 - m_0)\bar{c}},
\]
\[
\lambda = \frac{3(1 - m_0)^3(2 - \beta dA(m_0))}{M^2 \beta^2 \bar{d}^3 (A(m_0))^3},
\]
\[
\sigma = \frac{2(1 - m_0)^2(-3 + 2\beta dA(m_0))}{M \beta^2 \bar{d}^2 (A(m_0))^2}.
\]

(IV) Next, we check the optimality of the specified strategy.

(i) The optimality of the strategy of an agent with money holding \( \eta > 0 \):
First, we show that there exists a \( p(\eta) \geq p \) in (1). If \( \eta \in (0, \bar{p} \bar{q}] \), then by (6),
\[
aq + \beta V(\eta - p \bar{q}) = aq \left( 1 - \beta A(m_0) \frac{p_0}{p} \right) + a\beta A(m_0) \frac{\eta}{p}
\]
holds. Thus if
\[
1 - \beta A(m_0) \frac{p_0}{p} \geq 0
\]
holds, then she clearly chooses the maximum amount she can buy, and otherwise she chooses \( q_b = 0 \). Note that \( 1 - \beta A(m_0) \geq 0 \) holds, since \( \beta A(m_0) < 1 \). Let
\[
p(\eta) = \frac{p}{\beta A(m_0)}, \quad \text{for } \eta \in (0, \hat{p}q].
\]
Then, (1) is optimal for \( \eta \in (0, \hat{p}q] \). Moreover, \( p(\eta) \geq p \) clearly holds. Similar arguments apply to the case of \( \eta \in (\hat{p}q, \infty) \).

Next, we check an incentive for an agent with \( \eta > 0 \) to become a buyer instead of becoming a seller and offering \((p', q')\). By (1) and (13), for any \( p' > \frac{p}{\beta A(m_0)} \), no buyer accepts such an offer on the equilibrium, and then the value is the same as that of an offer \((p'', 0)\), where \( p'' \leq \frac{p}{\beta A(m_0)} \). Therefore, we can restrict our attention to \((p', q')\) such that \( p' \in \left[0, \frac{p}{\beta A(m_0)}\right] \) and \( q' \in [0, \bar{q}] \). By (1), the value of becoming a seller and offering \((p', q')\)
is
\[
\frac{1 - m_0}{k} \left[ -c + \int_{(0, \hat{p}q]} \beta \tilde{V}(\eta, \eta') \frac{f(\eta')}{1 - m_0} d\eta' \right] + \left( 1 - \frac{1 - m_0}{k} \right) \beta V(\eta),
\]
where
\[
\tilde{V}(\eta, \eta') = \begin{cases} V(\eta + \eta'), & \text{if } \eta' \leq p'q', \\ V(\eta + p'q'), & \text{if } \eta' > p'q'. \end{cases}
\]
On the other hand, when she becomes a buyer, the value is \( V(\eta) \). Thus the difference is
\[
\frac{1 - m_0}{k} \left[ -c + \int_{(0, \hat{p}q]} \beta \left( \tilde{V}(\eta, \eta') - V(\eta) \right) \frac{f(\eta')}{1 - m_0} d\eta' \right] - (1 - \beta)V(\eta).
\]
Below, we show
\[
V(\eta + \eta') - V(\eta) \leq aA(m_0) \frac{\eta'}{p}, \quad \text{for } \eta' \in (0, \hat{p}q].
\]
First, there exits a unique nonnegative integer \( n \) such that \( np\bar{q} \leq \eta < (n + 1)p\bar{q} \). There are two cases: (a) \( \eta + \eta' < (n + 1)p\bar{q} \) and (b) \( \eta + \eta' \geq (n + 1)p\bar{q} \). In case (a), by (6) and \( \beta A(m_0) < 1 \),
\[
V(\eta + \eta') - V(\eta) \leq \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^{n} \left[ \bar{q} - (1 - \beta A(m_0)) \eta + \epsilon - np\bar{q} \right] \right\}
- \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^{n} \left[ \bar{q} - (1 - \beta A(m_0)) \eta - np\bar{q} \right] \right\}
\leq aA(m_0) (\beta A(m_0))^{n} \frac{\eta'}{p}
\leq aA(m_0) \frac{\eta'}{p}.
\]
In case (b), by (6) and \( \beta A(m_0) < 1 \),

\[
V(\eta + \eta') - V(\eta) = \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^{n+1} \left[ \bar{q} - (1 - \beta A(m_0)) \frac{\eta + \eta' - (n + 1)p\bar{q}}{p} \right] \right\}
- \frac{aA(m_0)}{1 - \beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^n \left[ \bar{q} - (1 - \beta A(m_0)) \frac{\eta - np\bar{q}}{p} \right] \right\}
= aA(m_0) (\beta A(m_0))^n \left[ (1 - \beta A(m_0))(n + 1)\bar{q} + \beta A(m_0) \frac{\eta'}{p} - (1 - \beta A(m_0)) \frac{\eta}{p} \right]
\leq aA(m_0) (\beta A(m_0))^n \frac{\eta'}{p}
\leq aA(m_0) \frac{\eta'}{p}.
\]

The fourth line is obtained by \( \eta \geq (n + 1)p\bar{q} - \eta' \). This completes the proof of (16).

(6), (14), and (16) imply

\[\tilde{V}(\eta, \eta') - V(\eta) \leq aA(m_0) \frac{\eta'}{p}, \text{ for } \eta' \in (0, p\bar{q}].\]

Then, the first term of (15) is less than or equal to

\[
\frac{1 - m_0}{k} \left[ -\bar{c} + \int_{(0,p\bar{q}]} a\beta A(m_0) \frac{\eta'}{p} f(\eta') \frac{d\eta'}{1 - m_0} \right].
\]

This is equal to zero by the first equality of (7), and thus (15) is non-positive and she becomes a buyer.

(ii) The optimality of the strategy of an agent without money:

By the construction, an agent without money is indifferent between a buyer and a seller. Thus she has an incentive to be a seller. As in the latter part of (i), we restrict our attention to offers \((p', q')\) such that \(p' \in \left[ 0, \frac{p}{\beta A(m_0)} \right] \) and \(q' \in [0, \bar{q}] \). By (1) and (6), the value of offering \((p', q')\) is

\[
\frac{1 - m_0}{k} \left[ -\bar{c} + \int_{(0,p\bar{q}]} \beta \tilde{V}(0, \eta') \frac{f(\eta')}{1 - m_0} d\eta' \right], \quad (17)
\]

where \(\tilde{V}\) is defined in (14). If \(p'q' \geq p\bar{q}, \tilde{V}(0, \eta') = V(\eta')\) for any \(\eta' \in (0, p\bar{q}]\). Then, (17) is
the same for all \( p'q' \geq pq \), and therefore the offer \((p, \bar{q})\) is optimal. If \( p'q' \leq pq \),
\[
\int_{(0,pq]} \tilde{V}(0, \eta') f(\eta') d\eta' = \int_{(0,pq')} V(\eta') f(\eta') d\eta' + \int_{(pq',pq]} V(pq) f(\eta') d\eta'
\leq \int_{(0,pq')} V(\eta') f(\eta') d\eta' + \int_{(pq',pq]} V(\eta') f(\eta') d\eta'
= \int_{(0,pq]} \tilde{V}(0,pq) f(\eta') d\eta',
\]
where the inequality is obtained by (6) and (14). Then, the offer \((p, \bar{q})\) is optimal. This completes the proof of (IV).

(V) Finally, we check \( f(\eta) \geq 0 \) for all \( \eta \in (0, pq] \). Since \( f \) is linear, it suffices to show \( f(0) \geq 0 \) and \( f(pq) \geq 0 \). By (10), (11), and (12),
\[
f(0) = \sigma = \frac{2(1 - m_0)^2(-3 + 2\beta dA(m_0))}{M\beta^2d^2(A(m_0))^2}
\]
and
\[
f(pq) = 2\lambda pq + \sigma = \frac{2(1 - m_0)^2(3 - \beta dA(m_0))}{M\beta^2d^2(A(m_0))^2}
\]
hold. A sufficient condition for \( f(0) \geq 0 \) and \( f(pq) \geq 0 \) is clearly
\[
\frac{3}{2} \leq \beta dA(m_0) \leq 3.
\]
By the assumption \( d \leq 3 \), \( \beta dA(m_0) \leq 3 \) is always satisfied. It is easily verified that \( \frac{3}{2} \leq \beta dA(m_0) \) is equivalent to
\[
m_0 \geq \frac{3k(1 - \beta)}{3(2d - 3)}. \tag{18}
\]
Setting \( \beta = \frac{3k}{3(2d - 3)} \), we can show that for any \( \beta \in (\beta, 1) \) there exists a continuum of \( m_0 \) satisfying (18) and \( m_0 \in (0, 1) \). Indeed, \( \beta < 1 \) follows from the assumption \( d > \frac{3}{2} \), and \( 1 > \frac{3k(1 - \beta)}{3(2d - 3)} \) follows from \( \beta > \beta \).

The stationary condition is expressed as follows:
\[
m_0 \frac{1 - m_0}{k} \int_{(0,pq]} \frac{f(\eta')}{1 - m_0} d\eta' = \int_{(0,pq]} f(\eta') \frac{m_0}{k} \frac{m_0}{k}, \quad \text{for} \ \eta \in [0,pq],
\]
where the LHS is the outflow from \([0, \eta]\), while the RHS is the inflow into \([0, \eta]\). Nevertheless, this is automatically satisfied. This concludes the proof.
2 Proof of Theorem 3

Theorem 3 Suppose \( \frac{3}{2} < d \leq 3 \). Let \( \beta = \frac{3k}{3d-1+2d} \). Then, for any given \( \beta \in (\beta, 1) \) and \( B > B = \frac{3M\beta(2d-3)}{2\beta(2d-3)-3k(1-\beta)} \), there exists a continuum of stationary equilibria satisfying the commodity-money refinement in the sense of Zhou.

Proof:
Throughout the proof, we assume \( \epsilon \geq 0 \).

(I) We consider the same strategy and money holdings distribution as in KS: for some \( p \in (0, \frac{B}{\beta}) \),

- an agent without money always chooses to be a seller and an agent with money holding \( \eta > 0 \) always chooses to be a buyer,
- a seller always offers \( (p, \bar{q}) \),
- a buyer with money holding \( \eta > 0 \) consumes the following amount of her consumption good: there exists a \( p(\eta) \geq p \) such that, for given \( (p_s, q_s) \),
  \[
  q_b(\eta, p_s, q_s) = \begin{cases} 
  \min\{\eta/p_s, q_s\} & \text{if } p_s \leq p(\eta), \\
  0 & \text{if } p_s > p(\eta),
  \end{cases}
  \]

- for some \( \lambda \) and \( \sigma \), \( f \) is expressed by
  \[
  f(\eta) = \begin{cases} 
  2\lambda \eta + \sigma, & \text{for } \eta \in (0, \frac{B}{\beta}], \\
  0, & \text{for } \eta \in (\frac{B}{\beta}, \infty].
  \end{cases}
  \]

Note that \( p, p(\eta), \lambda, \) and \( \sigma \) will be determined as functions of \( m_0 \) later.

(II) Next, we obtain a candidate for a value function \( V : \mathbb{R}_+ \rightarrow \mathbb{R} \) consistent with the above strategy. From the above strategy, \( V(\eta) \) for \( \eta \in (0, B] \) can be written as a function of \( V(0) \) as follows. First, for \( \eta \in (0, \frac{B}{\beta}] \),

\[
V(\eta) = \epsilon \eta + \frac{m_0}{k} \left( a\frac{\eta}{p} + \beta V(0) \right) + \left( 1 - \frac{m_0}{k} \right) \beta V(\eta)
\]

holds. Thus
\[
V(\eta) = A(m_0) \left( a\frac{\eta}{p} + \beta V(0) \right) + Z(m_0)\eta, \quad \text{for } \eta \in (0, \frac{B}{\beta}],
\]

(21)
where \( A(m_0) = \frac{m_0}{k-(k-m_0)\beta} \) and \( Z(m_0) = \frac{k\epsilon}{k-(k-m_0)\beta} \). Note that \( A(m_0) < 1 \). Similarly, \( V(\eta) \) for \( \eta \in (p\bar{q}, B] \) is written as:

\[
V(\eta) = A(m_0)(a\bar{q} + \beta V(\eta - p\bar{q})) + Z(m_0)\eta, \quad \text{for} \; \eta \in (p\bar{q}, \infty).
\] (22)

Next, since an agent without money always chooses to be a seller, then \( V(0) \) is determined by

\[
V(0) = 1 - \frac{m_0}{k} c + \int_{(0,p\bar{q}]} \beta V(\eta) \frac{f(\eta)}{1-m_0 d\eta} + \left(1 - \frac{1-m_0}{k}\right) \beta V(0).
\] (23)

(III) Below, we focus on equilibria with \( V(0) = 0 \) and obtain \((p, \lambda, \sigma)\) as functions of \( m_0 \).

First, we decompose \( \eta \geq 0 \) into a multiple of \( p\bar{q} \) and a residual; that is, \( \eta = np\bar{q} + \iota \), where \( n \) is a nonnegative integer and \( \iota \) is a nonnegative real number less than \( p\bar{q} \). Then, by (21) and (22),

\[
V(np\bar{q} + \iota) = \frac{aA(m_0)}{1-\beta A(m_0)} \left\{ \bar{q} - (\beta A(m_0))^n \left[ \bar{q} - (1 - \beta A(m_0)) \frac{\iota}{p} \right] \right\} + Z(m_0) \left\{ n(1 - \beta A(m_0)) - \beta A(m_0) + (\beta A(m_0))^{n+1} \right\} \frac{1 - (\beta A(m_0))^{n+1} p\bar{q} + \frac{1}{2} (1 - \beta A(m_0))^{n+1}}{2} \}
\] (24)

holds. On the other hand, by (20) and (23),

\[
(1-m_0)c = \beta \left( \frac{aA(m_0)}{p} + Z(m_0) \right) \int_{(0,p\bar{q}]} \eta f(\eta) d\eta = \beta \left( \frac{aA(m_0)}{p} + Z(m_0) \right) \left( \frac{2}{3} \lambda p^3 q^3 + \frac{1}{2} \sigma p^2 q^2 \right)
\] (25)

holds.

Below, we obtain \((p, \lambda, \sigma)\) as functions of \( m_0 \). First, \( 1 - m_0 = \int_{(0,B]} f d\eta \) can be written as follows:

\[
1 - m_0 = \int_{(0,p\bar{q}]} f d\eta = \lambda p^2 q^2 + \sigma p\bar{q}.
\] (26)

Since the total amount of money the agents have is equal to \( M \), the following equation must be satisfied:

\[
M = \int_{(0,p\bar{q}]} \eta f d\eta = \frac{2}{3} \lambda p^3 q^3 + \frac{1}{2} \sigma p^2 q^2.
\] (27)
By (25), (26), (27), we obtain

\[p = \frac{Ma\beta A(m_0)}{(1 - m_0)c - M\beta Z(m_0)}, \quad (28)\]

\[\lambda = \frac{3(2M - p\bar{q}(1 - m_0))}{p^3\bar{q}^3}, \quad (29)\]

\[\sigma = \frac{2(-3M + 2p\bar{q}(1 - m_0))}{p^2\bar{q}^2}. \quad (30)\]

Suppose

\[\varepsilon < \frac{(1 - \beta)\bar{c}}{M\beta}. \quad (31)\]

Then, \(p \leq \frac{B}{q}\) is satisfied if and only if

\[m_0 \leq \bar{m}_0 = -\frac{1}{2} \left(-1 + \frac{k(1 - \beta)}{\beta} + \frac{Md}{B}\right) + \sqrt{\frac{1}{4} \left(-1 + \frac{k(1 - \beta)}{\beta} + \frac{Md}{B}\right)^2 + \frac{k(1 - \beta)}{\beta} - \frac{Mk}{\bar{c}} \varepsilon. \quad (32)\]

It is verified \(\bar{m}_0 \in (0, 1)\). We can also show that \(m_0 \leq \bar{m}_0\) implies \(p > 0\). Hereafter, we focus on \(m_0\) satisfying (32).

(IV) Next, we check the optimality of the specified strategy.

(i) The optimality of the strategy of an agent with money holding \(\eta > 0\):

First, we show that there exists a \(p(\eta) \geq p\) in (19). If \(\eta \in (0, p\bar{q}]\), then by (24),

\[aq + \beta V(\eta - psq) = \left(a - a\beta A(m_0)\frac{ps}{p} - \beta psZ(m_0)\right) q + \beta \left(\frac{aA(m_0)}{p} + Z(m_0)\right) \eta\]

holds. Thus if

\[a - a\beta A(m_0)\frac{ps}{p} - \beta psZ(m_0) \geq 0\]

holds, then she clearly chooses the maximum amount she can buy, and otherwise she chooses \(q_b = 0\). Let

\[p(\eta) = \frac{ap}{\beta(aA(m_0) + pZ(m_0))}, \quad \text{for } \eta \in (0, p\bar{q}]. \quad (33)\]

Then, (19) is optimal for \(\eta \in (0, p\bar{q}]\). Suppose

\[\varepsilon \leq \frac{(1 - \beta)\bar{c}(1 - \bar{m}_0)}{M\beta}, \quad (34)\]
then (28) and (32) imply that $p(\eta) \geq p$ holds. Hereafter, we focus on $\varepsilon$ satisfying (34). (31) clearly holds. Similar arguments apply to the case of $\eta \in (p\bar{q}, B]$.

Next, we check an incentive for an agent with $\eta > 0$ to become a buyer instead of becoming a seller and offering $(p', q')$. By (19) and (33), for any $p' > \frac{ap}{\beta(aA(m_0) + pZ(m_0))}$, no buyer accepts such an offer on the equilibrium, and then the value is the same as that of an offer $(p'', 0)$, where $p'' \leq \frac{ap}{\beta(aA(m_0) + pZ(m_0))}$. Therefore, we can restrict our attention to $(p', q')$ such that $p' \in \left[0, \frac{ap}{\beta(aA(m_0) + pZ(m_0))}\right]$ and $q' \in [0, \bar{q}](\text{provided } p'q' \leq B - \eta)$. By (19), the value of becoming a seller and offering $(p', q')$ is

$$
\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0,p\bar{q})} \beta \tilde{V}(\eta, \eta') \frac{f(\eta')}{1 - m_0} d\eta' \right] + \left(1 - \frac{1 - m_0}{k}\right) \beta V(\eta),
$$

where

$$
\tilde{V}(\eta, \eta') = \begin{cases} 
V(\eta + \eta'), & \text{if } \eta' \leq p'q', \\
V(\eta + p'q'), & \text{if } \eta' > p'q'.
\end{cases}
$$

(35)

On the other hand, when she becomes a buyer, the value is $V(\eta)$. Thus the difference is

$$
\frac{1 - m_0}{k} \left[-\bar{c} + \int_{(0,p\bar{q})} \beta \left(\tilde{V}(\eta, \eta') - V(\eta)\right) \frac{f(\eta')}{1 - m_0} d\eta' \right] - (1 - \beta)V(\eta).
$$

(36)

Below, we show

$$
V(\eta + \eta') - V(\eta) \leq \left(\frac{aA(m_0)}{p} + Z(m_0)\right) \eta', \text{ for } \eta' \in (0, p\bar{q}].
$$

(37)

First, there exits a unique nonnegative integer $n$ such that $np\bar{q} \leq \eta < (n + 1)p\bar{q}$. There are two cases: (a) $\eta + \eta' < (n + 1)p\bar{q}$ and (b) $\eta + \eta' \geq (n + 1)p\bar{q}$. In case (a), by (24), (28), (32), (34), and $\beta A(m_0) < 1$,

$$
V(\eta + \eta') - V(\eta) = \left[(\beta A(m_0))^n \left(\frac{aA(m_0)}{p} - \frac{\beta A(m_0)Z(m_0)}{1 - \beta A(m_0)}\right) + \frac{Z(m_0)}{1 - \beta A(m_0)}\right] \eta'
$$

$$
\leq \left(\frac{aA(m_0)}{p} + Z(m_0)\right) \eta'.
$$
In case (b), by (24), (28), (32), (34) and $\beta A(m_0) < 1$,

\[
V(\eta + \eta') - V(\eta) = (\beta A(m_0))^n \left\{ \left( aA(m_0) - \frac{p\beta A(m_0)Z(m_0)}{1 - \beta A(m_0)} \right) \right.
\]
\[
\times \left[ (n+1)(1 - \beta A(m_0))\bar{q} + \beta A(m_0) \frac{\eta'}{p} - (1 - \beta A(m_0)) \frac{\eta}{p} \right] \right\} + \frac{Z(m_0)}{1 - \beta A(m_0)} \eta'.
\]
\[
\leq \left[ (\beta A(m_0))^n \left( \frac{aA(m_0)}{p} - \frac{\beta A(m_0)Z(m_0)}{1 - \beta A(m_0)} \right) + \frac{Z(m_0)}{1 - \beta A(m_0)} \right] \eta'.
\]
\[
\leq \left( \frac{aA(m_0)}{p} + Z(m_0) \right) \eta'.
\]

The fourth line is obtained by $\eta \geq (n+1)p\bar{q} - \eta'$. This completes the proof of (37).

(24), (35), and (37) imply

$$
\bar{V}(\eta, \eta') - V(\eta) \leq \left( \frac{aA(m_0)}{p} + Z(m_0) \right) \eta', \quad \text{for } \eta' \in (0, p\bar{q}].
$$

Then, the first term of (36) is less than or equal to

$$
\frac{1 - m_0}{k} \left[ -\bar{c} + \int_{(0,p\bar{q}]} \left( \frac{aA(m_0)}{p} + Z(m_0) \right) \eta' \frac{f(\eta')}{1 - m_0} d\eta' \right].
$$

This is equal to zero by the first equality of (25), and thus (36) is non-positive and she becomes a buyer.

(ii) The optimality of the strategy of an agent without money:

By the construction, an agent without money is indifferent between a buyer and a seller. Thus she has an incentive to be a seller. As in the latter part of (i), we restrict our attention to offers $(p', q')$ such that $p' \in \left[ 0, \frac{ap}{aA(m_0) + pZ(m_0)} \right]$ and $q' \in [0, \bar{q}]$ (provided $p'q' \leq B$). By (19) and (24), the value of offering $(p', q')$ is

$$
\frac{1 - m_0}{k} \left[ -\bar{c} + \int_{(0,p\bar{q}]} \beta \bar{V}(0, \eta') \frac{f(\eta')}{1 - m_0} d\eta' \right],
$$

where $\bar{V}$ is defined in (35). If $p'q' \geq p\bar{q}$, $\bar{V}(0, \eta') = V(\eta')$ for any $\eta' \in (0, p\bar{q}]$. Then, (17) is
the same for all \( p'q' \geq p\bar{q} \), and therefore the offer \((p, \bar{q})\) is optimal. If \( p'q' \leq p\bar{q} \),

\[
\int_{(0,p\bar{q})} \tilde{V}(0, \eta') f(\eta') d\eta' = \int_{(0,p'q')} V(\eta') f(\eta') d\eta' + \int_{(p'q', p\bar{q})} V(\eta') f(\eta') d\eta' \
\leq \int_{(0,p'q')} V(\eta') f(\eta') d\eta' + \int_{(p'q', p\bar{q})} V(\eta') f(\eta') d\eta' \\
= \int_{(0,p\bar{q})} \tilde{V}(0, p\bar{q}) f(\eta') d\eta',
\]

where the inequality is obtained by (24) and (35). Then, the offer \((p, \bar{q})\) is optimal. This completes the proof of (IV).

(V) Finally, we check \( f(\eta) \geq 0 \) for all \( \eta \in (0, p\bar{q}] \). Since \( f \) is linear, it suffices to show \( f(0) \geq 0 \) and \( f(p\bar{q}) \geq 0 \). By (29) and (30),

\[
f(0) = \sigma = \frac{2(-3M + 2p\bar{q}(1 - m_0))}{p^2\bar{q}^2}
\]

and

\[
f(p\bar{q}) = 2\lambda p\bar{q} + \sigma = \frac{2(3M - p\bar{q}(1 - m_0))}{p^2\bar{q}^2}
\]

hold. A sufficient condition for \( f(0) \geq 0 \) and \( f(p\bar{q}) \geq 0 \) is clearly

\[
\frac{3}{2} \leq \frac{p\bar{q}(1 - m_0)}{M} \leq 3.
\]

By (28), this is equivalent to

\[
\frac{3}{2} \leq \frac{m_0(1 - m_0)\alpha\beta \bar{q}}{(1 - m_0)(k - (k - m_0)\beta)\bar{c} - M\beta k\varepsilon} \leq 3.
\]

By (32), (34), and the assumption \( d \leq 3 \), the second inequality is always satisfied. Also, by (34), the first inequality is satisfied if

\[
m_0 \geq \frac{3k(1 - \beta)}{\beta(2d - 3)}
\]

holds. Therefore, there exists a continuum of \( m_0 \) satisfying (32) and (39) whenever \( \bar{m}_0 \geq \frac{3k(1 - \beta)}{\beta(2d - 3)} \). By (32) and \( d \frac{3}{2} \), this is equivalent to

\[
\frac{1}{B} < \frac{2(\beta(2d - 3) - 3k(1 - \beta))}{3M\beta(2d - 3)}.
\]
Setting $\beta = \frac{3k}{3k-1+2d}$, and for any $\beta \in (\overline{\beta}, 1)$, setting $B = \frac{3M\beta(2d-3)}{2(3d-3k(1-\beta))}$, we can show that for any $\beta \in (\overline{\beta}, 1)$ and for any $B > B$, there exists a continuum of $m_0$ satisfying (32) and (39). Indeed, $\beta < 1$ follows from the assumption $d > \frac{3}{2}$.

The stationary condition is expressed as follows:

\[
m_0 \frac{1-m_0}{k} \int_{(\eta, p\bar{q})} \frac{f(\eta)}{1-m_0} d\eta' = \int_{(\eta, p\bar{q})} f(\eta') \frac{m_0}{k}, \quad \text{for } \eta \in [0, p\bar{q}],
\]

where the LHS is the outflow from $[0, \eta]$, while the RHS is the inflow into $[0, \eta]$. Nevertheless, this is automatically satisfied. This concludes that for any $\beta \in (\overline{\beta}, 1)$ and for any $B > B$, there exists a continuum of stationary equilibria if $\varepsilon$ is so small that (34) holds, which means that any stationary equilibrium satisfies the commodity-money refinement in the sense of Zhou.

References